Bosonic Quantum Computational Complexity

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16th Innovations in Theoretical Computer Science
January 13, 2025

What is this talk about

Main question: Study of computational complexity over continuous variable systems such as Bosonic systems.

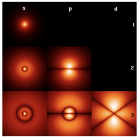
$$\text{Bosonic systems} = \text{Infinite-dim systems} = \text{Continous-variable systems}$$

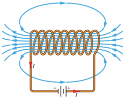
We consider two settings:

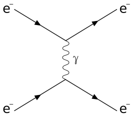
1. Dynamics
2. Ground state problem

Motivation

- Many physical systems are continuous variables.
  - Position and momenta
  - Oscillators (e.g., LC circuits)
  - E.M. amplitudes
  - Solving molecular structures
  - Q.F.T.







- Bosonic computational devices
- Computation over real numbers

Discrete v.s. continuous variables

DV	**CV**
Qbits/ Qdits (Spins, polarization)	Qmodes (Position, Momentum, Particle number)
Unit vectors in $\mathbb{C}^d$	Unit vectors in $\mathbb{C}^\infty$ (e.q. Square integrable functions)
Observables have discrete and bounded spectra	Observables may have bounded, unbounded, discrete or continuous spectra
Clifford/non-Clifford	Gaussian/non-Gaussian
Quantum complexity classes (BQP, QMA, etc)	??? (this talk)

Quantum states

- We can view quantum states in different basis
  - **Position basis:** $\psi \in \mathcal{L}^2(\mathbb{R})$

$$\psi(x), \quad \int_{\mathbb{R}} |\psi(x)|^2 dx = 1,$$

  or momentum basis via Fourier transform!
  - **Particle number (Fock) basis:**[1] $|\psi\rangle \in \ell^2(\mathbb{C})$

$$|\psi\rangle = \sum_{n\geq 0} \alpha_n |n\rangle\,, \quad \sum_{n\geq 0} |\alpha_n|^2 = 1$$

  where $\{|n\rangle\,, n \geq 0\}$ is known as the Fock or particle number basis.
- Coherent states

$$|z\rangle = e^{-|z|^2/2} \sum_{n\geq 0} \frac{z^n}{\sqrt{n!}} |n\rangle\,, \quad z \in \mathbb{C}$$

[1] $\psi(x) = \sum_{n\geq 0} \alpha_n H_n(x)$, H_n Hermite polynomials.

Unbounded operators

- **Position** (X) and **momentum** (P) have continuous eigenbasis

$$X|x\rangle = x|x\rangle, x \in \mathbb{R}, \quad P|p\rangle = p|p\rangle, p \in \mathbb{R}$$

  with algebra

$$[X, P] = iI$$

  They are Fourier dual!

- **Particle number:** $N = \frac{1}{2}(X^2 + P^2 - I)$ has discrete spectrum

$$N|n\rangle = n|n\rangle, n \geq 0.$$

- **Multiple modes:** O_j is operator O on mode j.

Gaussian gates

- Consider **polynomial hamiltonians** H of deg k in $(X_1, \dots, P_1, \dots)$, $H = H^\dagger$.
- $U = e^{iH}$ is a Gaussian unitary if hamiltonian H has deg ≤ 2 in X_i and P_j
  - Displacement $e^{i(aX+bP)}$
  - Rotation $e^{i\theta N}$
  - Squeezing $e^{ir(XP+PX)}$
  - Passive linear optical elements: Beam splitters and phase shifters.
- A quantum state $|\psi\rangle$ is called Gaussian if $|\psi\rangle = U|0\rangle$ for Gaussian U.
- A specific Gaussian gate is $F = e^{i\frac{\pi}{4}(X^2+P^2)}$ which is called Fourier transform because

$$FXF^{-1} = P, \quad FPF^{-1} = -X,$$

Gaussian gates

Gaussian operations are easy to represent:
If U is Gaussian

- **Single mode:**

$$U = \text{Rotation} \times \text{Displacement} \times \textit{Squeezing}$$

- **Multiple modes:**

$$U = V \bigotimes_i G_i W$$

V, W are passive linear optical and G_i are single mode Gaussian

Theorem

Starting with Gaussian states, Gaussian circuits can be efficiently simulated in polynomial time.

This can be viewed as a CV generalization of the Gottesman-Knill theorem for (DV) Clifford circuits.

Stellar rank

- For a CV quantum state $|\psi\rangle \in \mathbb{C}^\infty$, we can assign a Holomorphic "stellar" function

$$F_\psi(z) = e^{\frac{|z|^2}{2}} \langle z^* | \psi \rangle$$

  Normalization: $\int_{\mathbb{C}} |F_\psi(z)|^2 d\mu(z) = 1$, where $d\mu(z) = \frac{e^{-|z|^2}}{\pi} d^2 z$ is the Gaussian measure.

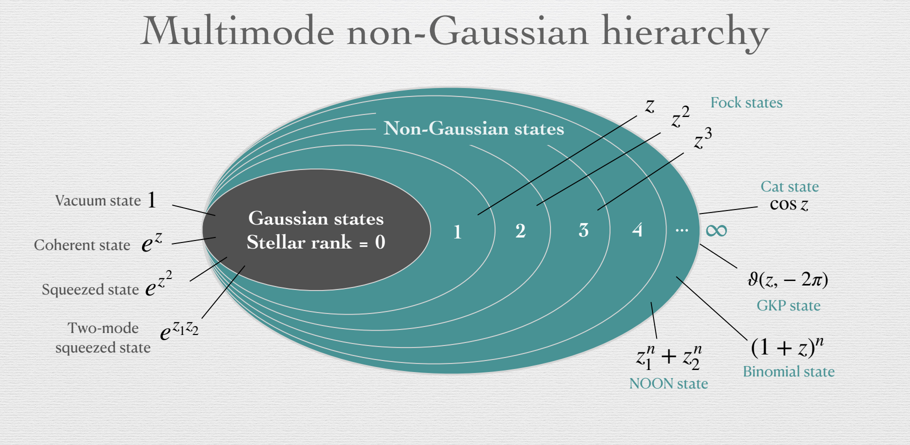
- Any stellar function can be decomposed as

$$F_\psi(z) = P_r(z) \times G(z).$$

  where $G(z)$ is a Gaussian function and P is a polynomial of degree $0 \leq r \leq \infty$

- r is called the stellar rank of $|\psi\rangle$. It measures "non-Gaussianity".

Multiple systems



Gaussian dynamics

We mentioned Gaussian computations can be simulated in polynomial time. But what is the precise complexity?

Definition

GDC is the class of decision problems that can be solved in Logspace if we can sample from Logspace uniform Gaussian computations.

Theorem (This work)

GDC $=$ **BQL**.

Proof idea: Reduce from inverting well-conditioned matrices.
This can be viewed as a generalization of a result of Aaronson and Gottesman that Clifford's computations are equivalent to $\oplus$**L**.

non-Gaussian dynamics

By adding a non-Gaussian gate such as $e^{iX^3/3}$, the model becomes very difficult to represent.

Definition

Let U be a unitary circuit of size s in cubic and Gaussian gates. Starting with the vacuum state:

1. **Amplitude estimation:** Estimate the probability of measuring the vacuum in the output.
2. **Observable estimation:** Estimate the expectation value of an observable with constant degree.

Due to the infinite-dimensional setting, it is not a priori clear whether we would get a decidable model.
For DV computations, both tasks can be done in complexity class $\#\mathbf{P}$.

non-Gaussian dynamics

We observe that under cubic gates observables **grow extremely fast**, but still **with a bounded rate**.

- $P \xrightarrow{X^3/3} P + X^2 \xrightarrow{F} -X + P^2 \xrightarrow{X^3/3} -X + (P + X^2)^2$
- If we continue this t rounds we get **repeated squaring**, i.e., the leading term would be X^{2^t}.
- The expected number of particles in the system may grow **doubly exponentially fast!**

Theorem (This work)

If U is a multi-mode circuit with t cubic gates with poly(t) size, then the **expected particle number** *of the quantum state $U|0\rangle$ is at most $2^{2^{O(t)}}$.*

This implies that the Hilbert space can be truncated to doubly exponential size, which implies an upper bound of **EEXP** on both tasks.

non-Gaussian dynamics

Can we do better? We can indeed:

Theorem (This work)

Amplitude estimation can be done in **EXPSPACE**. *Observable estimation can be done in* **PSPACE**.

Proof idea: We apply depth reduction techniques.
Question: Why can we do much better on the second task?

- Observable estimation can be viewed as estimating $\langle 0| U^{-1} O U |0\rangle$.
- We may hope to reduce amplitude est to observable est via $O = |0\rangle\langle 0|$
- $|0\rangle\langle 0|$ has an infinite degree in X and P, and it is not clear how one would cut it off without getting divergence.

The ground state problem

Let χ_r^N be the family of states with stellar rank $\leq r$ and energy $\leq N$.

Definition (CVLH$_r^E$)

Let H be a Hamiltonian of constant degree $k \geq 4$, let $E_r^N = Inf_{x \in \chi_r^N} \langle x| H |x\rangle$. Given the promise that either $E_r^N \geq a$ or $\leq a - 1/poly(|H|)$ decide which one is the case.

We show

Theorem (This work; Complexity of CVLH$_r^N$)

State family	*Complexity of CVLH*
$r = 0, N \leq exp$	**NP**-*complete*
$r, N \leq poly$	$\subseteq$ **QMA**
No bounds[a]	**RE**-*hard*

Furthermore the problem of deciding boundedness of Hamiltonianss is generally undecidable. When $k = 4$ and we restrict the family to Gaussians, it is **NP**-*hard.*

[a] $k \geq 8$

Open problems

- Lower bounds!
- CV Solovay Kitaev?
- Adding dissipation
- Implications to physics
- Establishing trade-offs between time, space, and precision.

Thank you!